Mathematics II

(English course)

Second semester, 2012/2013

Exercises (5)

1. Using the chain-rule, compute the following:

   (a) $\frac{df}{dt}$, with $f(x,y) = \frac{x+y}{1+y^2}$, $x = t\sin t$, $y = e^{t^2+1}$.

   (b) $\frac{dw}{dt}$, with $w = \frac{xy+y}{x^2+y^2}$, $x = \operatorname{tg} t$, $y = e^t$.

   (c) $\frac{df}{dt}$, with $f(u,v) = \ln\left(u+v^2\right)$, $u = \frac{x}{y}$, $v = xy$, $x = \frac{1}{t}$, $y = \frac{t}{t^2+1}$.

   (d) $\nabla g(-1,2)$, with $g(x,y) = \frac{ue^v}{v^2+w^2}$, $u = \frac{x}{y}$, $v = x^2+y^2$, $w = x^2-y^2$.

   (e) $\frac{\partial g}{\partial z}(0,1,-3)$, with $g(x,y,z) = \frac{(u-3v)^7}{1+w^4}$, $u = x^2+3xy$, $v = xy$, $w = (3y+z)^2$.

2. Write the second-order Taylor formula for the following functions:

   (a) $f(x,y) = \frac{xy}{x^2+y^2}$, at the point $(1,-1)$.

   (b) $f(x,y) = \frac{(x+1)y}{x^4+2y^4}$, at the point $(1,0)$.

   (c) $f(x,y) = \ln\frac{x+y^2}{1+xy}$, at the point $(2,1)$.

3. Let $f : \mathbb{R}^2 \mapsto \mathbb{R}$ be a twice continuously differentiable function, let $u : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be the function defined as

$$u(r,t) = (r\cos t, r\sin t)\,.$$

   Show that the function $g = f \circ u$ satisfies the equality

$$\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) \circ u = \frac{\partial^2 g}{\partial r^2} + \frac{1}{r}\frac{\partial g}{\partial r} + \frac{1}{r^2}\frac{\partial^2 g}{\partial t^2}$$

4. Consider a twice continuously differentiable function $f : \mathbb{R} \mapsto \mathbb{R}$, and let

$$u(x,y) = f(x-\alpha y) + f(x+\alpha y),$$

   where α is a real constant.

   Show that u solves the equation

$$\alpha^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}.$$

5. Consider a C_2-function, $f : \mathbb{R}^2 \mapsto \mathbb{R}$, and let

$$w = xf\left(xy, \frac{y}{x}\right).$$

(a) Show that

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} - w = 2x^2y\frac{\partial f}{\partial u}.$$

(b) Find $\frac{\partial^2 w}{\partial y^2}$.

6. Consider a differentiable function $f : \mathbb{R}^2 \mapsto \mathbb{R}$ such that

$$\|\nabla f(x,y)\| = 1, \qquad \forall (x,y) \in \mathbb{R}^2,$$

and let $g(u,v) = f(2uv, u^2 - v^2)$,

(a) Show that

$$\|\nabla g(u,v)\|^2 = 4(u^2 + v^2), \qquad \forall (u,v) \in \mathbb{R}^2.$$

(b) Assuming $\nabla f(2,0) = \left(\frac{3}{5}, \frac{-4}{5}\right)$, find $\nabla g(1,1)$.